General existence results for weight three nonlift paramodular cusp forms of prime level are due to Ibukiyama [7], who gave dimension formulae for weight three paramodular cusp forms of prime level, $S_3\left(K(p)\right)$. In conjunction with the known dimensions of spaces of Jacobi forms, one can see that the first weight three nonlifts with rational eigenvalues occur at levels $p = 61$, 73, and 79. These same levels had been identified in the work of Ash, Gunnells, and McConnell [1] as occurring in $H^5\left(\Gamma_0(N), \mathbb{C}\right)$; the congruence subgroup $\Gamma_0(N) \subseteq \mathrm{SL}_4(\mathbb{Z})$ is defined by having a bottom row in $(N\mathbb{Z}, N\mathbb{Z}, N\mathbb{Z}, \mathbb{Z})$. There is still no known mapping from nonlift paramodular cuspidal eigennewforms with rational eigenvalues into the cohomology space $H^5\left(\Gamma_0(N), \mathbb{C}\right)$, but perhaps one can be constructed using the orthogonal point of view. These paramodular cusp forms for $p \in \{61, 73, 79\}$ were directly constructed by Poor and Yuen in [13]. The eigenforms were constructed there as rational functions of Gritsenko lifts of theta blocks. Using this construction, the 2-, 3-, and 5-Euler factors were computed, integral Fourier expansions of content one were given, and congruences modulo ℓ for the Fourier expansions of these nonlift eigenforms to those of Gritsenko lifts were identified in each case. For $p = 61$, we have $\ell = 43$; for $p = 73$, we have $\ell = 3, 13$; and for $p = 79$, we have $\ell = 2$. In view of the results in [6], it is clear that these same constructions may be written as a sum of Gritsenko lifts and a Borcherds product.

Borcherds products have turned out to be a very useful tool for the construction of paramodular cusp forms. Indeed, there is an algorithm on the arXiv [12] for classifying all Borcherds products in all spaces $S_k\left(K(N)\right)$. The efficient implementation of this algorithm relies on a good knowledge of a determining number of Fourier–Jacobi coefficients for a given $S_k\left(K(N)\right)$, and the algorithm has already been successfully used in [2] and in [10]. Even very basic assertions, such as knowing that a certain paramodular form is an eigenform, hinge on a rigorous spanning set for $S_k\left(K(N)\right)$. To this end, upper and lower bounds for $\dim S_k\left(K(N)\right)$ must be computed separately. For squarefree N there is a dimension formula, due to Ibukiyama and Kitayama [8]. For N not squarefree, the upper bounds are approachable via the method of *Jacobi restriction* [9, 3, 11], which classifies possible initial Fourier–Jacobi expansions of paramodular cusp forms. Lower bounds, on the other hand, require the construction of paramodular forms by techniques such as Borcherds products and Hecke spreading of Borcherds products, compare [11].

Once a relevant space of paramodular forms has been spanned, typically because there is a matching arithmetic object as a candidate for modularity, one can broach the separate question of computing enough Euler factors to rigorously prove this modularity. In weight two, such a strategy has recently had success in [4] for proving examples of modularity for typical abelian surfaces defined over $\mathbb{Q}$. The number of Euler factors that can be computed is sensitive to the manner of construction of the paramodular eigenform, but good results can be expected for paramodular eigenforms which are constructed as rational functions of Gritsenko lifts of theta blocks. One natural arithmetic candidate for modularity in the weight three case is given by the class of hypergeometric motives.

Dave Roberts sent Poor, Shurman, and Yuen (henceforth PSY) some hypergeometric motives with motivic Galois group GSp(4). One, for example, had conductor $N = 257$, and PSY did find a new form with rational eigenvalues in $S_3\,(K(257))$, whose 2-Euler factor in the arithmetic normalization was

$$1 + x + 6x^2 + 8x^3 + 64x^4.$$

This matched the 2-Euler factor that Dave Roberts had. The paramodular Atkin–Lehner sign here was -1. Such hypergeometric motives are good candidates for examples of rigorous modularity proofs following the pattern of [4]. Indeed, the theory of Galois representations is better understood in weight three than in weight two, and the existence of dimension formulae for prime level (and at least conjecturally for squarefree level) makes the weight three case look more approachable than the weight two case. PSY have recently written a number of new programs aimed at gathering computational data, both rigorous and heuristic, about weight three paramodular cusp forms. However, the dimensions of the weight three spaces grow more quickly than in weight two.

Also, Henri Cohen, in his *Computing L-functions: A survey* [5], says that the Dwork quintic pencil

$$x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 - 5\psi x_1x_2x_3x_4x_5, \qquad (\psi \in \mathbb{Q})$$

can give a hypergeometric motive of conductor $N = 525$, which should be modular with respect to a paramodular newform in $S_3\,(K(N))$. So far, however, no arithmetic geometer has presented any Euler factors for this case. Of course, the level $N = 525$ is not squarefree, but such examples have in principle been dealt with in [10], where $N = 16$ was successfully considered.

Due the start-up costs of beginning a computation, the best manner in which to proceed is for PSY to respond to motives with known conductors and Euler factors, trying to locate paramodular newforms that match them. If anyone wants to send PSY such a target, we will start working on that case to provide at least heuristic information.

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